Table 3 Robustness results for two PPF filter combination

PPF modal frequency, Hz	First mode decibel drop (% increase)	Second mode decibel drop (% increase)
$\omega_{c1} = 1.3 \times 1.0 = 1.3$ $\omega_{c2} = 7.1 \times 1.0 = 7.1$	58.33 (513%)	44.00 (97%)
$\omega_{c1} = 1.3 \times 1.25 = 1.625$ $\omega_{c2} = 7.1 \times 1.25 = 8.875$	42.16 (343%)	38.10 (70%)
$\omega_{c1} = 1.3 \times 1.5 = 1.95$ $\omega_{c2} = 7.1 \times 1.5 = 10.65$	36.14 (280%)	36.35 (62%)
$\omega_{c1} = 1.3 \times 1.75 = 2.275$ $\omega_{c2} = 7.1 \times 1.75 = 12.45$	30.11 (216%)	32.99 (47%)
$\omega_{c1} = 1.3 \times 2 = 2.6$ $\omega_{c2} = 7.1 \times 2 = 14.2$	17.88 (88%)	26.57 (19%)

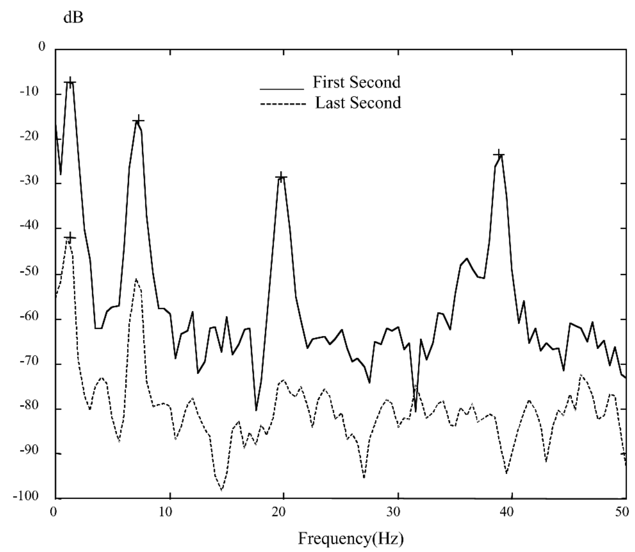

Fig. 5 PSD plot for two PPF filters (compensator frequencies are 1.5 times the targeted modal frequencies.).

damping effect. Just as in the single PPF case, the PPF combination shows good robustness for both modes. These results suggest that along with being robust the two PPF filter combination is effective in damping multiple modes over a range of frequencies. A PSD plot for the same controller graphically illustrating damping effectiveness is shown in Fig. 5.

VII. Conclusions

This research presents the experimental results robustness study of vibration suppression of a flexible structure using PPF control. The flexible structure is a cantilevered beam with PZT sensors and PZT actuators. PPF controls were implemented for single-mode vibration suppression and for multimode vibration suppression. Experiments found that PPF control is robust to frequency variations for single-mode and for multimode vibration suppressions.

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Approximate Equations for the Coplanar Restricted Three-Body Problem

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Introduction

THE fuel-optimal rendezvous problem for a spacecraft with a satellite in a central force field has been the subject of several papers [1–6] in the last decade. In this Note we address the rendezvous problem of a spacecraft with a moon, comet, or asteroid (whose gravitational field is not negligible) in the central force field of a (large) third body. In this setting we derive reduced equations for the motion of the spacecraft in the vicinity of the smaller body (moon) in three dimensions. We present a generalization, under some restrictions, to the Jacobi integral.[7]

Clearly the problem we consider here is closely related to the restricted three-body problem in two dimensions,[7,8] which deals with the motion of a body of negligible mass in the gravitational field of two other celestial bodies. However in this setting all three bodies are assumed to remain always in one plane.

To put our results in proper perspective, we note that Edelbaum[9] made one of the first attempts to address the rendezvous problem between a spacecraft and a satellite in a near circular orbit. A simpler model was derived by Clohessy and Wiltshire,[10] whose equations can be found now in books on orbital mechanics.[4] Carter and Humi,[1] Humi,[2] and Carter and Brient[3] found some analytical solutions for the rendezvous problem when the satellite is in a general Keplerian orbit and derived a generalization[2] of these equations in the presence of a general central force field. Currently an effort is under way to include the effect of drag (linear and quadratic) in the rendezvous equations.[6] (For a more extensive list of contributions to the rendezvous problem, see the reference lists in Refs. 1 and 6).

Derivation of the Reduced Equations

In an inertial coordinate system whose origin is at the center of the central body E, we let $\boldsymbol{R}$, $\boldsymbol{\rho}$ denote the position of the moon and the spacecraft, respectively, and $\boldsymbol{r}$ the relative position of the

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spacecraft with respect to the moon. We then have $\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r}$, and the equation of motion of the spacecraft is given by

$$m_s \ddot{\boldsymbol{\rho}} = -\left(GM_E m_s \big/ \rho^3\right)\boldsymbol{\rho} - \left(GM_m m_s \big/ r^3\right)\boldsymbol{r} + \boldsymbol{F}_s \tag{1}$$

In this equation $\boldsymbol{F}_s$ is the force exerted by the spacecraft thrusters; $\rho = |\boldsymbol{\rho}|$; $r = |\boldsymbol{r}|$; M_E, M_m, m_s are the masses of the central body, moon, and spacecraft, respectively; G is the constant of gravity; and dots denote differentiation with respect to time. Assuming that $r \ll R$ (the spacecraft is in the vicinity of the moon) and using

$$\rho = (\boldsymbol{\rho}\cdot\boldsymbol{\rho})^{\frac{1}{2}} = (\boldsymbol{R}+\boldsymbol{r}, \boldsymbol{R}+\boldsymbol{r})^{\frac{1}{2}} = R(1 - 2\boldsymbol{R}\cdot\boldsymbol{r}/R^2 + r^2/R^2)^{\frac{1}{2}} \tag{2}$$

we can (by Taylor expansion) make the following approximation[1] to Eq. (1):

$$\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} = \left(-GM_E \big/ R^3\right)[\boldsymbol{R} + \boldsymbol{r} - (3\boldsymbol{R}\cdot\boldsymbol{r}/R^2)\boldsymbol{R}] - \left(GM_m \big/ r^3\right)\boldsymbol{r} + \boldsymbol{a}_3 \tag{3}$$

where $\boldsymbol{a}_s = \boldsymbol{F}_s/m_s$. Using the fact that at any point

$$\ddot{\boldsymbol{R}} = -\left(GM_E \big/ R^3\right)\boldsymbol{R} \tag{4}$$

Eq. (3) reduces to

$$\ddot{\boldsymbol{r}} = -\left(GM_E \big/ R^3\right)[\boldsymbol{r} - (3\boldsymbol{R}\cdot\boldsymbol{r}/R^2)\boldsymbol{R}] - \left(GM_m \big/ r^3\right)\boldsymbol{r} + \boldsymbol{a}_s \tag{5}$$

From the law of angular momentum conservation, we have

$$M_m R^2 \Omega = L = \text{const} \tag{6}$$

where $\boldsymbol{\Omega} = (\mathrm{d}\theta/\mathrm{d}t)\boldsymbol{e}_z$. Using the relation (6) to eliminate R^3 from Eq. (5), we obtain

$$\ddot{\boldsymbol{r}} = -\frac{GM_E M_m^{\frac{3}{2}}}{L^{\frac{3}{2}}}\Omega^{\frac{3}{2}}\left(\boldsymbol{r} - \frac{3\boldsymbol{R}\cdot\boldsymbol{r}}{R^2}\boldsymbol{R}\right) - \frac{GM_m}{r^3}\boldsymbol{r} + \boldsymbol{a}_s \tag{7}$$

In a coordinate system rotating with the moon (and fixed at its center), Eq. (7) becomes

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega}\times\boldsymbol{r} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\boldsymbol{r} = k\Omega^{\frac{3}{2}}\left(\boldsymbol{r} - \frac{3\boldsymbol{R}\cdot\boldsymbol{r}}{R^2}\boldsymbol{R}\right) - \frac{GM_m}{r^3}\boldsymbol{r} + \boldsymbol{a}_s \tag{8}$$

where $k = GM_E M_m^{3/2}/L^{3/2}$. In this rotating system we now let the x axis be transverse but opposed to the motion of the moon, the y axis in the direction of $\boldsymbol{R}$ and the z axis completes a right-handed system. In this frame $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{R} = (0, 1, 0)R$, and $\boldsymbol{\Omega} = (0, 0, \dot{\theta}) = (0, 0, \omega)$. In component form Eq. (8) then becomes

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x - \dot{\omega} y = -k\omega^{\frac{3}{2}}x - GM_m x\big/r^3 + a_1 \tag{9}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y + \dot{\omega} x = -k\omega^{\frac{3}{2}}y + 3k\omega^{\frac{3}{2}}y - GM_m y\big/r^3 + a_2 \tag{10}$$

$$\ddot{z} = -k\omega^{\frac{3}{2}}z - GM_m z\big/r^3 + a_3 \tag{11}$$

where $\boldsymbol{a}_s = (a_1, a_2, a_3)$. We now perform a change of variables from t to $\theta(t)$ in these equations and divide the resulting equations by ω. This leads to

$$\omega x'' + \omega' x' = \left(\omega - k\omega^{\frac{1}{2}}\right)x + \omega' y + 2\omega y' - \alpha x/r^3 + \bar{a}_1 \tag{12}$$

$$\omega y'' + \omega' y' = -\omega' x + \left(\omega + 2k\omega^{\frac{1}{2}}\right)y - 2\omega x' - \alpha y/r^3 + \bar{a}_2 \tag{13}$$

$$\omega z'' + \omega' z' = -k\omega^{\frac{1}{2}}z - \alpha z/r^3 + \bar{a}_3 \tag{14}$$

where $\alpha = GM_m/\omega$, $\bar{a}_i = a_i/\omega$ and primes denote derivatives with respect to θ.

Because the moon is in a Keplerian orbit around E, we have[4]

$$1/R = C(1 + e\cos\theta), \qquad \omega = \left(LC^2\big/M_m\right)(1 + e\cos\theta)^2 \tag{15}$$

where $C = GM_E M_m^2/L^2$.

Substituting these expressions in Eq. (10) and dividing by $LC^2/M_m(1+\cos\theta)$ yields after some algebra

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 x']' = \left[(1+e\cos\theta) - \frac{kM_m^{\frac{1}{2}}}{CL^{\frac{1}{2}}}\right]x + 2[(1+e\cos\theta)y] - \frac{GM_m^3 x}{L^2C^4(1+e\cos\theta)^3 r^3} + \bar{\bar{a}}_1 \tag{16}$$

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 y']' = \left[(1+e\cos\theta) + \frac{2kM_m^{\frac{1}{2}}}{L^{\frac{1}{2}}C}\right]y - 2[(1+e\cos\theta)x]' - \frac{GM_m^3 y}{L^2C^4(1+e\cos\theta)^3 r^3} + \bar{\bar{a}}_2 \tag{17}$$

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 z']' = -\frac{kM_m^{\frac{1}{2}}}{L^{\frac{1}{2}}C}z - \frac{GM_m^3 z}{L^2C^4(1+e\cos\theta)r^3} = \bar{\bar{a}}_3 \tag{18}$$

where $\bar{\bar{a}}_i = M_m a_i/LC^2(1+e\cos\theta)$.

We now introduce the variables

$$u = (1+e\cos\theta)x, \qquad v = (1+e\cos\theta)y, \qquad w = (1+e\cos\theta)z \tag{19}$$

Substituting these variables in Eq. (12) leads to

$$u'' = \frac{u}{1+e\cos\theta}\left(1 - \frac{kM_m^{\frac{1}{2}}}{CL^{\frac{1}{2}}}\right) + 2v' - \frac{Au}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_1 \tag{20}$$

$$v'' = \frac{v}{1+e\cos\theta}\left(1 + 2\frac{kM_m^{\frac{1}{2}}}{CL^{\frac{1}{2}}}\right) - 2u' - \frac{Av}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_2 \tag{21}$$

$$w'' = -w - \frac{Aw}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_3 \tag{22}$$

where $\sigma^2 = u^2 + v^2 + w^2$ and $A = GM_m^3/L^2C^4$. However, it is now easy to see from the definition of K, L, C that

$$kM_m^{\frac{1}{2}}\big/CL^{\frac{1}{2}} = 1 \tag{23}$$

and hence the equations of motion reduce to

$$u'' = 2v' - \frac{Au}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_1 \tag{24}$$

$$v'' = \frac{3v}{1+e\cos\theta} - 2u' - \frac{Av}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_2 \tag{25}$$

$$w'' = -w - \frac{Aw}{(1+e\cos\theta)\sigma^3} + \bar{\bar{a}}_3 \tag{26}$$

In this form the equations of motion resemble closely those that we derived for the rendezvous of a spacecraft and a satellite in Ref. 1. Furthermore, in spite of the fact that these equations are nonlinear they are easily amenable to numerical computations. We observe also that if $A \ll 1$ then it is straightforward to obtain approximate solutions to these equations by first-order perturbation expansion in A. (Because the solution of these equations with $A = 0$ is well known[1]).

For the special case $e = 0$ and $\bar{\bar{a}}_i = 0$, it is possible to derive a first integral of these equations. This can be done by multiplying Eqs. (24–26) by u', v', w', respectively, and summing. We obtain

$$[(u')^2 + (v')^2 + (w')^2]' = (3v^2 - w^2)' - (A/\sigma^3)(\sigma^2)' \tag{27}$$

Integrating this equation (with respect to θ) we then obtain

$$(u')^2 + (v')^2 + (w')^2 - 3v^2 + w^2 - 2A/\sigma = \text{constant} \tag{28}$$

This first integral can be considered as an "extension" of the Jacobi first integral[7] for the restricted three-body problem in two dimensions.

Finally we would like to consider the two-dimensional restricted three-body problem (i.e., the spacecraft trajectory is in the $x - y$ plane) with $e = 0$ and $a_i = 0$.

To treat this problem, we introduce

$$u = \sigma \cos\phi, \qquad v = \sigma \sin\phi \tag{29}$$

Equation (28) then becomes

$$(\sigma')^2 + \sigma^2(\phi')^2 - 2A/\sigma = 3\sigma^2 \sin^2\phi \tag{30}$$

Furthermore, by differentiating $\sigma^2 = u^2 + v^2$ twice and using Eqs. (24–26) and (28) we obtain

$$\sigma\sigma'' + (\sigma')^2 - 2\sigma^2\phi' - A/\sigma = 6\sigma^2 \sin^2\phi \tag{31}$$

It is easy to verify that (as in the classical case) the system (30), (31) has an (implicit) solution when $\phi(\theta)$ = constant. Moreover we can reduce Eqs. (30) and (31) to a first-order system by changing the independent variable from θ to σ and introduce $p = \mathrm{d}\sigma/\mathrm{d}\theta$ as a new dependent variable. After some algebra we obtain

$$p^2\left[1 + \sigma^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2\right] - \frac{2A}{\sigma} = 3\sigma^2 \sin^2\phi \tag{32}$$

$$p\left(\sigma\frac{\mathrm{d}p}{\mathrm{d}\sigma} - p\right) = 2\sigma^2 p\left[p\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 + \frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right] - \frac{3A}{\sigma} \tag{33}$$

Conclusions

The equations derived in this Note for the restricted three-body problem in three dimensions closely resemble those that were derived by Carter and the author in a recent paper. They will be amenable to treatment by analytic and computational techniques. In this Note we considered only the equations of motion in a gravitational force field. However, we would like to conjecture that similar equations can be derived for the same system in a general central force field with the presence of linear drag as was done in the case of a spacecraft and a satellite.

As to the accuracy of the approximate rendezvous equations, we observe that we made two approximations in their derivation. The first results from neglecting the spacecraft's influence on the motion of the other two bodies in the system. The second is from the linearization of the equations of motion under the assumption $r \ll R$.

From a practical point of view, the first approximation is well suited for all celestial applications. The second becomes more accurate as $r/R \to 0$. Thus, unless the two celestial bodies are extremely close to each other the linearization of the equations of motion is well justified.

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Energy-Based Stabilization of Angular Velocity of Rigid Body in Failure Configuration

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I. Introduction

THE problem of stabilization of the angular velocity of a rigid body, modeling a simple satellite, has been addressed by several authors and is discussed and solved in this Note from a new perspective. From a control theoretic point of view the most interesting and studied case is the one of a body operating in failure configuration, that is, with only one or two independent actuators acting on the system. More precisely, in Refs. 1–4 it was shown that the zero solution of Euler's angular velocity equations can be made asymptotically stable by means of two control torques, whereas in Refs. 5–8 the same problem has been addressed and solved in the case of only one control torque. Robust stabilization has been studied in Refs. 9 and 10, and stabilization using partial state information has been addressed in Ref. 11. Finally, the problem of stabilization of nonzero (relative) equilibria has been studied in Refs. 11–13. In almost all of the aforementioned papers (Refs. 3, 12, and 13 are notable exceptions) the stabilization problem has been addressed and solved without making use of the special structure of the system to be controlled, that is, stabilizing control laws have been derived using various techniques, such as backstepping, center manifold theory, homogeneity considerations, and control Lyapunov functions

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